

## Exercise 6

Show that if  $\operatorname{Re} z_1 > 0$  and  $\operatorname{Re} z_2 > 0$ , then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2,$$

where principal arguments are used.

### Solution

Suppose that  $\operatorname{Re} z_1 > 0$  and  $\operatorname{Re} z_2 > 0$ . Then  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$  lie in the first or fourth quadrants:

$$-\frac{\pi}{2} < \theta_1 + 2n_1\pi < \frac{\pi}{2} \quad (1)$$

and

$$-\frac{\pi}{2} < \theta_2 + 2n_2\pi < \frac{\pi}{2}, \quad (2)$$

where  $n_1 = 0, \pm 1, \pm 2, \dots$  and  $n_2 = 0, \pm 1, \pm 2, \dots$ . We have

$$\arg z_1 = \arg r_1 e^{i\theta_1} = \theta_1 + 2n_1\pi.$$

Since this quantity on the right is between  $-\pi/2$  and  $\pi/2$ ,  $\arg z_1$  can be replaced by the principal argument  $\operatorname{Arg} z_1$ . (The requirement to use  $\operatorname{Arg} z$  is that  $-\pi < \operatorname{Arg} z \leq \pi$ .)

$$\arg z_1 = \operatorname{Arg} z_1$$

Similarly, we have

$$\arg z_2 = \arg r_2 e^{i\theta_2} = \theta_2 + 2n_2\pi.$$

Since this quantity on the right is between  $-\pi/2$  and  $\pi/2$ ,  $\arg z_2$  can be replaced by the principal argument  $\operatorname{Arg} z_2$ .

$$\arg z_2 = \operatorname{Arg} z_2$$

Add each of the sides of inequalities (1) and (2).

$$-\pi < \theta_1 + \theta_2 + 2n_1\pi + 2n_2\pi < \pi \quad (3)$$

We also have

$$\begin{aligned} \arg(z_1 z_2) &= \arg z_1 + \arg z_2 \\ &= \arg r_1 e^{i\theta_1} + \arg r_2 e^{i\theta_2} \\ &= (\theta_1 + 2n_1\pi) + (\theta_2 + 2n_2\pi) \\ &= \theta_1 + \theta_2 + 2n_1\pi + 2n_2\pi. \end{aligned}$$

Since

$$-\pi < \arg(z_1 z_2) < \pi,$$

$\arg(z_1 z_2)$  can be replaced by the principal argument  $\operatorname{Arg}(z_1 z_2)$ . Therefore, because

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2,$$

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

for the case that  $z_1$  and  $z_2$  have positive real components.